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On the Approximation of Positive Functions by Power Series, II

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We consider positive functions $h = h(x)$ defined for $x \in R_0^+$. Conditions for the existence of a power series $N(x) = \sum c_n x^n$, $c_n \geqslant 0$, with the property

$$d_1 \leqslant h(x)/N(x) \leqslant d_2, \qquad x \geqslant 0,$$

for some constants $d_1, d_2 \in R^+$, are investigated in [J. Clunie and T. Kövari, *Canad. J. Math.* **20** (1968), 7–20; P. Erdős and T. Kövari, *Acta Math. Acad. Sci. Hung.* **7** (1956), 305–316; U. Schmid, *Complex Variables* **18** (1992), 187–192; U. Schmid, *J. Approx. Theory* **83** (1995), 342–346]. In this paper, methods are discussed which allow for a given function h the construction of the coefficients c_n, $n \in \mathbb{N}_0$, for the above defined power series N and to find suitable constants d_1 and d_2. We also study the power series $H(x) = \sum x^n/u_n$, where we set $u_n = \sup\{x^n/h(x),\ x \geqslant 0\}$, for $n \in \mathbb{N}_0$, and the relation between h and H concerning the above stated inequalities.

1. INTRODUCTION

Let $h\colon R_0^+ \to R^+$ be rapidly growing, i.e.,

$$x^n/h(x) \to 0 \qquad \text{for} \quad x \to \infty, \quad \forall n \in \mathbb{N}. \tag{1}$$

The study of the question of whether there exists a power series $N(x) = \sum c_n x^n$, $c_n \geqslant 0$, with

$$d_1 \leqslant h(x)/N(x) \leqslant d_2, \qquad x \geqslant 0, \tag{2}$$

for some constants $d_1, d_2 \in R^+$ (notation $h \sim N$), is of particular interest in approximation problems [1, 2, 5–8] and in numerical applications. By the premise $c_n \geqslant 0$, $n \in \mathbb{N}_0$, the power series N has some elementary properties like monotonicity or convexity, but none of them are presumed by h. The only reason to require of h condition (1) is to exclude the trivial case, where $N(x)$ represents a polynomial.

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A solution to problem (2) stated above can be formulated with the aid of the function

$$P(x) = \max\{x^n/u_n, n \in \mathbb{N}_0\},$$

where we set $u_n = \sup\{x^n/h(x), x \geqslant 0\}$, $\forall n \in \mathbb{N}_0$.

In [10] the author proved the following

THEOREM 1. *Let $h: R_0^+ \to R^+$ be rapidly growing. A necessary and sufficient condition for the existence of a power series, $N(x) = \sum c_n x^n$, $c_n \geqslant 0$, with $h \sim N$ is the validity of*

$$h(x)/P(x) \leqslant c, \qquad x \in R_0^+, \tag{3}$$

for some constant $c \in R^+$.

The initial intention of this article is to demonstrate methods for the construction of the power series $N(x)$ and to find constants $d_1, d_2 \in R^+$ which satisfy (2). We shall give a solution to this problem in the proof of Theorem 4 by means of the sequence $u_n = \sup\{x^n/h(x), x \geqslant 0\}$, $n \in \mathbb{N}_0$.

On the other hand there is a natural connection between the function $P(x) = \max\{x^n/u_n, n \in \mathbb{N}_0\}$ and the power series $H(x) = \sum x^n/u_n$. It is easy to prove that $(u_n)^{1/n} \to \infty$ for $n \to \infty$; hence the function $H(x)$ is well defined for every $x \in R$. One would expect that for an arbitrary function h satisfying (3) the corresponding power series H solves our approximation problem (2), i.e., $h \sim H$. However, we shall see that this is not true for every h satisfying (3) and that the validity of $h \sim H$ depends mainly on the growth of $h(x)$ for $x \to \infty$. Nevertheless we give a complete criterion in solving this problem with Theorems 2 and 3, respectively.

2. THE POWER SERIES $H(x) = \sum x^n/u_n$

For a given positive function h we denote with A_n, $n \in \mathbb{N}_0$, the set of functions $\{y(x) \leqslant h(x),\ y(x) = ax^n \text{ and } a \in R^+\}$ and we set $y_n(x) = x^n/u_n$, where $u_n = \sup\{x^n/h(x),\ x \geqslant 0\}$. Then we have with $y_n \in A_n$ the maximal function of A_n which approximates or rather reaches h from below. This approximation, however, is of interest only in a finite interval, because for every $n \in \mathbb{N}_0$ we obtain, as a consequence of (1), that $h(x)/y_n(x) \to \infty$ for $x \to \infty$. Now we can ask if it is possible to extend this approximation to the positive number axis by virtue of the power series $H(x) = \sum y_n(x) = \sum x^n/u_n$. We discuss this question in the following

THEOREM 2. *Let* $h\colon R_0^+ \to R^+$ *be rapidly growing and let it satisfy* (3). *We set* $v_n = u_{n+1}/u_n$, $\forall n \in \mathbb{N}_0$. *A necessary and sufficient condition for*

$$h \sim H \tag{4}$$

is the existence of a positive constant $d < 1$ *and an integer* z *such that*

$$v_n \leqslant d v_{n+z}, \qquad \forall n \in \mathbb{N}_0. \tag{5}$$

Proof of Theorem 2. *Sufficiency*. The sequence $\{v_n, n \in \mathbb{N}_0\}$ is increasing because $u_{n+1}^2 = (\sup\{x^{n+1}/h(x), x \geqslant 0\})^2 = \sup\{(x^{n+2}/h(x)) * (x^n/h(x)), x \geqslant 0\} \leqslant \sup\{x^{n+2}/h(x), x \geqslant 0\} * \sup\{x^n/h(x), x \geqslant 0\} = u_{n+2}u_n$, i.e., $v_{n+1} = u_{n+2}/u_{n+1} \geqslant u_{n+1}/u_n = v_n$.

From this we obtain for every $n \in \mathbb{N}_0$

$$v_n^{z+1} u_n/u_{n+z+1} = \prod_{i=0}^{z} v_n/v_{n+i} \leqslant v_n/v_{n+z} \leqslant d,$$

and

$$v_{n+z+1}^{-z-1} u_{n+z+1}/u_n = \prod_{i=0}^{z} v_{n+i}/v_{n+z+1} \leqslant v_n/v_{n+z} \leqslant d.$$

With $s = z+1$ it follows that

$$v_n^s u_n/u_{n+s} \leqslant d \tag{6}$$

and

$$v_{n+s}^{-s} u_{n+s}/u_n \leqslant d, \qquad \forall n \in \mathbb{N}_0. \tag{7}$$

Now let us formulate two lemmas in advance.

LEMMA 1. *For* $i, n \in \mathbb{N}_0$, $x \in R_0^+$, *and*

(a) *for* $i \leqslant n$ *and* $x \geqslant v_{n-1}$ *or*

(b) *for* $i \geqslant n$ *and* $x \leqslant v_n$ *we have*

$$x^i/u_i \leqslant x^n/u_n.$$

Proof. Condition (a) in connection with the increasing of the sequence $\{v_n, n \in \mathbb{N}_0\}$ implies that

$$x^{n-i} \geqslant v_{n-1}^{n-i} \geqslant \prod_{k=1}^{n-i} v_{n-k} = \prod_{k=1}^{n-i} u_{n-k+1}/u_{n-k} = u_n/u_i, \qquad \text{i.e.,} \quad x^i/u_i \leqslant x^n/u_n.$$

Part (b) of Lemma 1 follows in the same way. ∎

An immediate consequence of Lemma 1 is that the function $P(x) = \max\{x^n/u_n, n \in \mathbb{N}_0\}$ has the following representation for $x \geq v_0$

$$P(x) = x^n/u_n, \qquad x \in [v_{n-1}, v_n), \quad n \in \mathbb{N}. \tag{8}$$

LEMMA 2(a). *For $m, k \in \mathbb{N}$ and $x \in [v_{(m-1)s}, v_{ms})$ we have*

$$(x^{ms+ks}/u_{ms+ks})/P(x) \leq d^k.$$

Proof. By the definition of P it follows that

$$P(x) \geq x^{ms}/u_{ms}, \qquad \text{for} \quad x \in [v_{(m-1)s}, v_{ms}). \tag{9}$$

Consequently, by (6) and (9), we obtain

$$\begin{aligned}(x^{ms+ks}/u_{ms+ks})/P(x) &\leq (x^{ms+ks}/u_{ms+ks})/(x^{ms}/u_{ms}) = x^{ks}u_{ms}/u_{ms+ks} \\ &\leq v_{ms}^{ks}u_{ms}/u_{ms+ks} = \prod_{i=1}^{k} v_{ms}^{s}u_{ms+(i-1)s}/u_{ms+is} \\ &\leq \prod_{i=1}^{k} v_{ms+(i-1)s}^{s}u_{ms+(i-1)s}/u_{ms+is} \leq d^k. \quad \blacksquare\end{aligned}$$

LEMMA 2(b). *For $m \geq 3$ and $x \in [v_{(m-1)s}, v_{ms})$ we have*

$$(x^{ms-ks}/u_{ms-ks})/P(x) \leq d^{k-1}, \qquad \textit{where} \quad 2 \leq k \leq m.$$

Proof. Again by the definition of P it follows that

$$P(x) \geq x^{ms-s}/u_{ms-s}, \qquad \text{for} \quad x \in [v_{(m-1)s}, v_{ms}). \tag{10}$$

Hence, by (7) and (10), we obtain

$$\begin{aligned}(x^{ms-ks}/u_{ms-ks})/P(x) &\leq (x^{ms-ks}/u_{ms-ks})/(x^{ms-s}/u_{ms-s}) \\ &= x^{s-ks}u_{ms-s}/u_{ms-ks} \leq v_{ms-s}^{s-ks}u_{ms-s}/u_{ms-ks} \\ &= \prod_{i=1}^{k-1} v_{ms-s}^{-s}u_{ms-is}/u_{ms-(i+1)s} \\ &\leq \prod_{i=1}^{k-1} v_{ms-is}^{-s}u_{ms-is}/u_{ms-(i+1)s} \leq d^{k-1}. \quad \blacksquare\end{aligned}$$

Using Lemma 1(b) for $0 \leq i \leq s-1$, $k \geq 1$, and $0 \leq x \leq v_{ms}$ we obtain

$$x^{ms+ks+i}/u_{ms+ks+i} \leq x^{ms+ks}/u_{ms+ks},$$

and consequently

$$\sum_{i=0}^{s-1} x^{ms+ks+i}/u_{ms+ks+i} \leqslant s x^{ms+ks}/u_{ms+ks}. \tag{11}$$

In the same way, using Lemma 1(a) for $0 \leqslant i \leqslant s-1$, $2 \leqslant k < m$, and $x \geqslant v_{ms-s}$, we get

$$\sum_{i=0}^{s-1} x^{ms-ks-i}/u_{ms-ks-i} \leqslant s x^{ms-ks}/u_{ms-ks}. \tag{12}$$

Finally, by (11), (12), and Lemma 2, we have for every $m \geqslant 3$ and $x \in [v_{(m-1)s}, v_{ms})$

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^n/u_n\right)\Big/ P(x) \\
&\quad = \left(1/u_0 + \sum_{n=1}^{ms-2s} x^n/u_n + \sum_{n=ms-2s+1}^{ms+s-1} x^n/u_n + \sum_{n=ms+s}^{\infty} x^n/u_n\right)\Big/ P(x) \\
&\quad = \left(\sum_{k=2}^{m-1}\sum_{i=0}^{s-1} x^{ms-ks-i}/u_{ms-ks-i} + 1/u_0 + \sum_{n=ms-2s+1}^{ms+s-1} x^n/u_n \right. \\
&\qquad \left. + \sum_{k=1}^{\infty}\sum_{i=0}^{s-1} x^{ms+ks+i}/u_{ms+ks+i}\right)\Big/ P(x) \\
&\quad \leqslant \left(s\sum_{k=2}^{m-1} x^{ms-ks}/u_{ms-ks} + 3sP(x) + s\sum_{k=1}^{\infty} x^{ms+ks}/u_{ms+ks}\right)\Big/ P(x) \\
&\quad \leqslant s\sum_{k=2}^{m-1} d^{k-1} + 3s + s\sum_{k=1}^{\infty} d^k \\
&\quad \leqslant s\sum_{k=1}^{\infty} d^k + 3s + s\sum_{k=1}^{\infty} d^k \\
&\quad = s(3-d)/(1-d) = (z+1)(3-d)/(1-d).
\end{aligned}$$

For $m=2$ or $m=1$ the above estimation follows in the same way, using only (11) and Lemma 2(a).

In the interval $x \in [0, v_0)$ we have $P(x) = 1/u_0 = P(v_0)$; therefore the inequality

$$\left(\sum_{n=0}^{\infty} x^n/u_n\right)\Big/ P(x) \leqslant (z+1)(3-d)/(1-d) \tag{13}$$

holds for every $x \geqslant 0$.

Comparing (13) and our assumption (3), we obtain the desired inequalities

$$(1-d)/((z+1)(3-d)) \leqslant h(x) \bigg/ \left(\sum_{n=0}^{\infty} x^n/u_n \right) \leqslant c, \qquad \text{for} \quad x \in R_0^+.$$

Necessity. Let us assume that condition (5) is not satisfied by h. Then for every integer s we can find an index $n_S \in \mathbb{N}_0$ with

$$v_{n_s}/v_{n_s+s} > (1/2)^{1/s}.$$

For $i \leqslant s$ we have

$$v_{n_s}/v_{n_s+i} \geqslant v_{n_s}/v_{n_s+s} > (1/2)^{1/s}.$$

Hence we get for every $s \in \mathbb{N}$

$$v_{n_s}^s u_{n_s}/u_{n_s+s} = \prod_{i=0}^{s-1} v_{n_s}/v_{n_s+i} > 1/2. \tag{14}$$

On the other hand we have for every $n \in \mathbb{N}_0$ and $k \geqslant 2$

$$v_n^k u_n/u_{n+k} = \prod_{i=0}^{k-1} v_n/v_{n+i} \leqslant \prod_{i=0}^{k-2} v_n/v_{n+i} = v_n^{k-1} u_n/u_{n+k-1}. \tag{15}$$

Using (14) and (15) we obtain for every $s \in \mathbb{N}$

$$\begin{aligned}\left(\sum_{n=0}^{\infty} v_{n_s}^n/u_n \right) \bigg/ P(v_{n_s}) &\geqslant \left(\sum_{k=1}^{s} v_{n_s}^{n_s+k}/u_{n_s+k} \right) \bigg/ (v_{n_s}^{n_s}/u_{n_s}) \\ &= \sum_{k=1}^{s} v_{n_s}^k u_{n_s}/u_{n_s+k} > s/2.\end{aligned}$$

Comparing the above estimation with our assumption (3) it follows that $h(v_{n_s})/\sum_{n=0}^{\infty} v_{n_s}^n/u_n \leqslant 2c/s \to 0$, for $s \to \infty$, which is in contradiction to (4).

COROLLARY. *Let h: $R_0^+ \to R^+$ be rapidly growing and let it satisfy* (3).

(a) *If there is a positive constant $d < 1$ with*

$$v_n/v_{n+1} \leqslant d, \qquad \forall n \in \mathbb{N}_0,$$

then we have

$$(1-d)/2(3-d) \leqslant h(x) \bigg/ \sum_{n=0}^{\infty} x^n/u_n \leqslant c, \qquad \textit{for} \quad x \in R_0^+.$$

(b) *If $v_n/v_{n+1} \to 1$, for $n \to \infty$, then $h \sim H$ is not satisfied.*

Proof. (a) follows directly from Theorem 2 with $z=1$.

(b) Let $d<1$ be an arbitrary positive constant. If we can find for every $s\in\mathbb{N}$ an index $n_s\in\mathbb{N}_0$ with $v_{n_s}/v_{n_s+s}>d$, then condition (4) is not satisfied by h.

The sequence v_n/v_{n+1} is convergent, i.e., $\lim v_n/v_{n+1}=1$. Hence there exists for every $s\in\mathbb{N}$ an index $n_s\in\mathbb{N}_0$ with

$$v_n/v_{n+1}>d^{1/s}, \qquad \forall n\geqslant n_s, \tag{16}$$

and therefore with (16) it follows that

$$v_{n_s}/v_{n_s+s}=\prod_{i=0}^{s-1} v_{n_s+i}/v_{n_s+i+1}>(d^{1/s})^s=d. \quad ∎$$

EXAMPLE. We consider the function $h(x)=x^{b\ln x}$, for $x\geqslant 1$ and an arbitrary constant $b>0$. Then we have $h\sim H$, or more precisely

$$(1-e^{-1/2b})/2(3-e^{-1/2b})\leqslant x^{b\ln x}\Big/\sum_{n=0}^{\infty} x^n e^{-n^2/4b}\leqslant e^{1/16b}, \qquad \text{for} \quad x\geqslant 1.$$

First of all the function $h(x)=x^{b\ln x}$ is investigated in [9] for $b=1$, where we get

$$x^{\ln x}/P(x)\leqslant e^{1/16}, \qquad \text{for} \quad x\geqslant 1.$$

In the same way we get for an arbitrary $b>0$

$$h(x)/P(x)\leqslant e^{1/16b}, \qquad \text{for} \quad x\geqslant 1,$$

i.e., the function $h(x)=x^{b\ln x}$ satisfies condition (3), for $x\geqslant 1$, with the constant $c=e^{1/16b}$.

On the other hand we have $u_n=\sup\{x^n/h(x), x\geqslant 1\}=e^{n^2/4b}$. Consequently we get $v_n=u_{n+1}/u_n=e^{n/2b-1/4b}$ and $v_n/v_{n+1}=e^{-1/2b}$. Hence, by virtue of our corollary, it follows with $d=e^{-1/2b}$ that

$$(1-e^{-1/2b})/2(3-e^{-1/2b})\leqslant x^{b\ln x}\Big/\sum_{n=0}^{\infty} x^n e^{-n^2/4b}\leqslant e^{1/16b}, \qquad \text{for} \quad x\geqslant 1.$$

Now let us turn to the question of whether there exist positive functions h with an arbitrarily strong growth and satisfying $h\sim H$. We shall see that our example stated above represents a natural limit of growth for all functions h satisfying $h\sim H$. This will be proved in

THEOREM 3. *Let $h: R_0^+ \to R^+$ be rapidly growing with $h \sim H$. Then there exist constants $a > 1$ and $b > 0$ with*

$$h(x) \leqslant x^{b \ln x}, \qquad \textit{for} \quad x \geqslant a.$$

Proof of Theorem 3. We set for $t \geqslant \ln v_0$, $f(t) = \ln P(e^t)$, i.e., $f(t) = mt - \ln u_m$, for $t \in [\ln v_{m-1}, \ln v_m)$, $m \in \mathbb{N}$. By virtue of Theorem 2 there exist a positive constant $d < 1$ and an integer z with $v_n \leqslant d v_{n+z}$, $\forall n \in \mathbb{N}_0$. We define for every $n \in \mathbb{N}_0$, $t_n = \ln v_0 + np$, where $p = -\ln d$. It follows that

$$p \leqslant \ln v_{n+z} - \ln v_n, \qquad \forall n \in \mathbb{N}_0. \tag{17}$$

Based on the above definition of the function f we can find for every $n \in \mathbb{N}$ an index m_n with

$$f(t_n) = m_n t_n - \ln u_{m_n}, \tag{18}$$

and

$$\ln v_{m_n - 1} \leqslant t_n < \ln v_{m_n}. \tag{19}$$

Using (17) and (19) we obtain $\ln v_{nz} = \ln v_0 + \sum_{i=1}^{n} (\ln v_{iz} - \ln v_{(i-1)z}) \geqslant \ln v_0 + np \geqslant \ln v_{m_n - 1}$, i.e., $v_{nz} \geqslant v_{m_n - 1}$, and therefore we have

$$m_n \leqslant nz + 1. \tag{20}$$

By (18), (20), and the convexity of f we have for every $n \in \mathbb{N}$ $(f(t_n) - f(t_{n-1}))/p = (f(t_n) - f(t_{n-1}))/(t_n - t_{n-1}) \leqslant f'_l(t_n) \leqslant m_n \leqslant nz + 1$, i.e.,

$$f(t_n) - f(t_{n-1}) \leqslant p(nz + 1). \tag{21}$$

Using (21) we obtain for every constant $s > 0$, $t \geqslant s$, and $t \in [t_{n-1}, t_n)$

$$\begin{aligned}
f(t) &= f(t_0) + \sum_{i=1}^{n-1} (f(t_i) - f(t_{i-1})) + f(t) - f(t_{n-1}) \\
&\leqslant f(t_0) + \sum_{i=1}^{n} (f(t_i) - f(t_{i-1})) \leqslant f(t_0) + \sum_{i=1}^{n} p(iz + 1) \\
&= f(\ln v_0) + pzn(n+1)/2 + pn \\
&\leqslant f(\ln v_0) + pz((t - \ln v_0)/p + 1)((t - \ln v_0)/p + 2)/2 + p((t - \ln v_0)/p + 1) \\
&\leqslant rt^2,
\end{aligned}$$

where $r>0$ is a suitable constant which depends on s, p, z, and v_0. Hence we get

$$P(x)\leqslant x^{r\ln x}, \qquad \text{for} \quad x\geqslant e^s. \tag{22}$$

From our assumption $h\sim H$ it follows, in view of Theorem 1, that there exists a constant $c>0$ with $h(x)/P(x)\leqslant c$, for $x\geqslant 0$. Comparing this result with inequality (22) we obtain for suitable constants $a>1$ and $b>0$, $h(x)\leqslant x^{b\ln x}$, for $x\geqslant a$. ∎

In some mathematical disciplines even power series with non-negative coefficients are of special importance. An example of this kind is the theory of orthogonal polynomials for weights on the real line. D. S. Lubinsky [6, 7] introduces for a weight $h(x)=e^{Q(x)}$, where Q is even and convex, the following power series

$$G_Q(x)=1+\sum_{n=1}^{\infty}(x/q_n)^{2n}\,e^{2Q(q_n)}n^{-1/2}$$

with $q_n^n e^{-Q(q_n)}=\max\{x^n e^{-Q(x)},\, x\geqslant 0\}$. Accordingly defined is $G_{Q/2}(x)$ (see also V. Totik [11]).

Using Laplace's method he demonstrates that

$$G_Q(x)=\sqrt{\pi T(x)}\,e^{2Q(x)}(1+O(Q(x)^{-1/2}(\ln x)^{-3/2})), \qquad x\to\infty,$$

where $T(x)=1+xQ''(x)/Q'(x)$.

This type of result is useful in the above-mentioned theory. The power series G_Q and $G_{Q/2}$ can be expressed in terms of the sequence $u_n=\sup\{x^n/h(x),\, x\geqslant 0\}$ as

$$G_Q(x)=1+\sum_{n=1}^{\infty}(x^n/u_n)^2\,n^{-1/2}$$

and

$$G_{Q/2}(x)=1+\sum_{n=1}^{\infty}(x^{2n}/u_{2n})\,n^{-1/2}, \quad \text{respectively.}$$

As a direct consequence of Theorem 2 we are able to formulate a necessary and sufficient condition for $h(x)\sim\sum x^{2n}/u_{2n}$, namely, that the asymptotic relation $h(x)\sim\sum x^{2n}/u_{2n}$ is synonymous with $h(\sqrt{x})\sim\sum x^n/u_{2n}$ and, using Theorem 2, with

$$w_n\leqslant dw_{n+z}, \qquad \forall n\in\mathbb{N}_0,$$

where $w_n=u_{2(n+1)}/u_{2n}$ $(\sup\{x^n/h(\sqrt{x}),\, x\geqslant 0\}=\sup\{x^{2n}/h(x),\, x\geqslant 0\}=u_{2n})$.

More generally we get in the same way conditions for $h(x) \sim \sum x^{kn}/u_{kn}$ for an arbitrary $k \in \mathbb{N}$.

3. THE CONSTRUCTION OF A POWER SERIES $N(x) = \sum c_n x^n$, $c_n \geqslant 0$, WITH $h \sim N$

Theorems 2 and 3 show that the asymptotic relation $h \sim H$ is not given for every function h satisfying (3). But at least we have the existence of a power series $N(x) = \sum c_n x^n$, $c_n \geqslant 0$, with $h \sim N$. We shall be concerned with the construction of this power series in the following

THEOREM 4. *Let $h\colon R_0^+ \to R^+$ be rapidly growing and let it satisfy* (3). *Then for every positive $d < 1$ we can select a subsequence $\{u_{n_k}, k \in \mathbb{N}_0\}$ from the sequence $\{u_n, n \in \mathbb{N}_0\}$ with*

$$(1-d)/4 \leqslant h(x) \Big/ \sum_{k=0}^{\infty} x^{n_k}/u_{n_k} \leqslant c/d, \qquad \textit{for} \quad x \geqslant 0.$$

Proof of Theorem 4. To construct the desired subsequence $\{u_{n_k}, k \in \mathbb{N}_0\}$ we formulate the next

LEMMA 3. *We define for every $n, s \in \mathbb{N}_0$*

$$A(n, s) = u_n v_n^s / u_{n+s} \qquad \textit{and} \qquad B(n, s) = u_{n+s} v_{n+s}^{-s} / u_n.$$

Then we have

(a) $A(n, s+1) \leqslant A(n, s)$ *and* $B(n, s+1) \leqslant B(n, s)$.

(b) $\lim_{s \to \infty} A(n, s) = \lim_{s \to \infty} B(n, s) = 0$.

(c) $A(n, s) \leqslant 1$ *and* $B(n, s) \leqslant 1$.

Proof. (a) The sequence v_n is increasing (proof of Theorem 2); hence for $s \geqslant 1$ we have

$$A(n, s) = \prod_{i=0}^{s-1} v_n/v_{n+i} \geqslant \prod_{i=0}^{s} v_n/v_{n+i} = A(n, s+1),$$

and

$$B(n, s) = \prod_{i=0}^{s-1} v_{n+i}/v_{n+s} \geqslant \prod_{i=0}^{s} v_{n+i}/v_{n+s+1} = B(n, s+1).$$

(b) To verify that $v_n \to \infty$, for $n \to \infty$, we assume the contrary $v_n \leqslant c$, $\forall n \in \mathbb{N}_0$, for some constant $c > 0$. Then we have

$$u_n \leqslant c^n u_0, \qquad \forall n \in \mathbb{N}_0. \tag{i}$$

On the other hand we get

$$u_n = \sup\{x^n/h(x), x \geqslant 0\} \geqslant (2c)^n/h(2c). \tag{ii}$$

Comparing (i) and (ii) it follows that $(2c)^n/h(2c) \leqslant c^n u_0$, i.e., $2^n \leqslant u_0 h(2c)$, $\forall n \in \mathbb{N}_0$, which is impossible.

As a consequence of $\lim_{n \to \infty} v_n = \infty$ we obtain for every $n \in \mathbb{N}_0$

$$A(n, s) = \prod_{i=0}^{s-1} v_n/v_{n+i} \leqslant v_n/v_{n+s-1} \to 0, \qquad \text{for} \quad s \to \infty,$$

and in the same way $\lim_{s \to \infty} B(n, s) = 0$.

(c) follows immediately from part (a) of this lemma and the fact that $B(n, 0) = A(n, 0) = 1$. ▮

Now we are able to define for every positive $d < 1$ a subsequence $\{u_{n_k}, k \in \mathbb{N}_0\}$ of the sequence $\{u_n, n \in \mathbb{N}_0\}$ with the desired property described in Theorem 4.

Put $u_{n_0} = u_0$. We define $u_{n_{k+1}}$, supposing that u_{n_k} is already defined, as follows: Let us denote with $s_k \in \mathbb{N}$ the smallest integer with

$$A(n_k, s_k) < d \tag{23}$$

and

$$B(n_k, s_k) < d. \tag{24}$$

Due to Lemma 3 this choice of $s_k \in \mathbb{N}$ is always possible and therefore at least one of the following inequalities (25, 26) holds,

$$A(n_k, s_k - 1) \geqslant d \tag{25}$$

or

$$B(n_k, s_k - 1) \geqslant d. \tag{26}$$

We define $u_{n_{k+1}} = u_{n_k + s_k - 1}$. Since $A(n_k, 1) = 1$ we have $s_k \geqslant 2$, $\forall k \in \mathbb{N}_0$.

In connection with (23) and Lemma 3(a) it follows that

$$u_{n_k} v_{n_k}^{n_{k+2}-n_k}/u_{n_{k+2}} = A(n_k, n_{k+2}-n_k) = A(n_k, s_k+s_{k+1}-2) \leqslant A(n_k, s_k) < d. \tag{27}$$

Using (24) and Lemma 3(a) we get

$$u_{n_{k+2}} v_{n_{k+2}}^{n_k-n_{k+2}}/u_{n_k} = B(n_k, n_{k+2}-n_k) = B(n_k, s_k+s_{k+1}-2) \leqslant B(n_k, s_k) < d. \tag{28}$$

With the notation $P_1(x) = \max\{x^{n_k}/u_{n_k}, k \in \mathbb{N}_0\}$, for $x \geqslant 0$, we shall prove the following

LEMMA 4. (a) *For $m \geqslant k \geqslant 3$ and $x \in [v_{n_{m-1}}, v_{n_m})$ we have*

$$(x^{n_{m-k}}/u_{n_{m-k}})/P_1(x) \leqslant \begin{cases} d^{k/2-1}, & \text{for even } k, \\ d^{(k-1)/2}, & \text{for odd } k. \end{cases}$$

(b) *For $m \in \mathbb{N}$, $k \geqslant 2$ and $x \in [v_{n_{m-1}}, v_{n_m})$ we have*

$$(x^{n_{m+k}}/u_{n_{m+k}})/P_1(x) \leqslant \begin{cases} d^{k/2}, & \text{for even } k, \\ d^{(k-1)/2}, & \text{for odd } k. \end{cases}$$

Proof. First of all we set for every $m \in \mathbb{N}_0$ $y_{n_m}(x) = x^{n_m}/u_{n_m}$. Then P_1 has the following representation in the interval $[v_{n_{m-1}}, v_{n_m})$

$$P_1(x) = \begin{cases} y_{n_{m-1}}(x), & \text{for} \quad x \in [v_{n_{m-1}}, w_m), \\ y_{n_m}(x), & \text{for} \quad x \in [w_m, v_{n_m}), \end{cases}$$

where $w_m = (u_{n_m}/u_{n_{m-1}})^{1/(n_m - n_{m-1})}$ represents, for every $m \in \mathbb{N}$, the x-coordinate of the intersection-point of $y_{n_{m-1}}(x)$ and $y_{n_m}(x)$.

Hence we have for $x \in [v_{n_{m-1}}, v_{n_m})$

$$P_1(x) \geqslant y_{n_{m-1}}(x) \qquad \text{and} \qquad P_1(x) \geqslant y_{n_m}(x). \tag{29}$$

To prove Lemma 4(a) we get by (29)

$$\begin{aligned}(x^{n_{m-k}}/u_{n_{m-k}})/P_1(x) &\leqslant (x^{n_{m-k}}/u_{n_{m-k}})/(x^{n_{m-1}}/u_{n_{m-1}}) \\ &= u_{n_{m-1}} x^{n_{m-k}-n_{m-1}}/u_{n_{m-k}} \\ &\leqslant u_{n_{m-1}} v_{n_{m-1}}^{n_{m-k}-n_{m-1}}/u_{n_{m-k}}.\end{aligned}$$

For even $k \geqslant 4$ and using (28) we have

$$
\begin{aligned}
&u_{n_{m-1}} v_{n_{m-1}}^{n_{m-k}-n_{m-1}} / u_{n_{m-k}} \\
&\quad = u_{n_{m-1}} v_{n_{m-1}}^{n_{m-2}-n_{m-1}} / u_{n_{m-2}} \prod_{i=1}^{k/2-1} u_{n_{m-2i}} v_{n_{m-1}}^{n_{m-2(i+1)}-n_{m-2i}} / u_{n_{m-2(i+1)}} \\
&\quad \leqslant \prod_{i=1}^{k/2-1} u_{n_{m-2i}} v_{n_{m-2i}}^{n_{m-2(i+1)}-n_{m-2i}} / u_{n_{m-2(i+1)}} < d^{k/2-1},
\end{aligned}
$$

where we use for the first inequality the fact that $u_{n_{m-1}} v_{n_{m-1}}^{n_{m-2}-n_{m-1}} / u_{n_{m-2}} = B(n_{m-2}, n_{m-1}-n_{m-2}) \leqslant 1$ (Lemma 3(c)). For odd $k \geqslant 3$ and using (28)

$$
\begin{aligned}
u_{n_{m-1}} v_{n_{m-1}}^{n_{m-k}-n_{m-1}} / u_{n_{m-k}} &= \prod_{i=0}^{(k-3)/2} u_{n_{m-2i-1}} v_{n_{m-1}}^{n_{m-2i-3}-n_{m-2i-1}} / u_{n_{m-2i-3}} \\
&\leqslant \prod_{i=0}^{(k-3)/2} u_{n_{m-2i-1}} v_{n_{m-2i-1}}^{n_{m-2i-3}-n_{m-2i-1}} / u_{n_{m-2i-3}} \\
&< d^{(k-1)/2}.
\end{aligned}
$$

Now let us prove Lemma 4(b), where we have by (29)

$$
\begin{aligned}
(x^{n_{m+k}}/u_{n_{m+k}})/P_1(x) \leqslant (x^{n_{m+k}}/u_{n_{m+k}})/(x^{n_m}/u_{n_m}) &= u_{n_m} x^{n_{m+k}-n_m} / u_{n_{m+k}} \\
&\leqslant u_{n_m} v_{n_m}^{n_{m+k}-n_m} / u_{n_{m+k}}.
\end{aligned}
$$

For even $k \geqslant 2$ and using (27) we have

$$
\begin{aligned}
u_{n_m} v_{n_m}^{n_{m+k}-n_m} / u_{n_{m+k}} &= \prod_{i=0}^{k/2-1} u_{n_{m+2i}} v_{n_m}^{n_{m+2(i+1)}-n_{m+2i}} / u_{n_{m+2(i+1)}} \\
&\leqslant \prod_{i=0}^{k/2-1} u_{n_{m+2i}} v_{n_{m+2i}}^{n_{m+2(i+1)}-n_{m+2i}} / u_{n_{m+2(i+1)}} < d^{k/2}.
\end{aligned}
$$

Finally for odd $k \geqslant 3$ and again using (27) we get

$$
\begin{aligned}
&u_{n_m} v_{n_m}^{n_{m+k}-n_m} / u_{n_{m+k}} \\
&\quad = u_{n_m} v_{n_m}^{n_{m+1}-n_m} / u_{n_{m+1}} \prod_{i=1}^{(k-1)/2} u_{n_{m+2i-1}} v_{n_m}^{n_{m+2i+1}-n_{m+2i-1}} / u_{n_{m+2i+1}} \\
&\quad \leqslant \prod_{i=1}^{(k-1)/2} u_{n_{m+2i-1}} v_{n_{m+2i-1}}^{n_{m+2i+1}-n_{m+2i-1}} / u_{n_{m+2i+1}} < d^{(k-1)/2},
\end{aligned}
$$

considering that by virtue of Lemma 3(c) we have $u_{n_m} v_{n_m}^{n_{m+1}-n_m} / u_{n_{m+1}} = A(n_m, n_{m+1}-n_m) \leqslant 1$. ∎

As a consequence of Lemma 4(a) we get for every $m \geqslant 3$ and $x \in [v_{n_{m-1}}, v_{n_m})$

$$\left(\sum_{k=3}^{m} x^{n_{m-k}}/u_{n_{m-k}} \right) \Big/ P_1(x) \leqslant 2 \sum_{i=1}^{\infty} d^i. \tag{30}$$

To verify (30) let us consider the case where m is an even integer. Then we have

$$\begin{aligned} &\left(\sum_{k=3}^{m} x^{n_{m-k}}/u_{n_{m-k}} \right) \Big/ P_1(x) \\ &\quad = \left(\sum_{i=1}^{m/2-1} x^{n_{m-2i-1}}/u_{n_{m-2i-1}} \right) \Big/ P_1(x) + \left(\sum_{i=2}^{m/2} x^{n_{m-2i}}/u_{n_{m-2i}} \right) \Big/ P_1(x) \\ &\quad \leqslant \sum_{i=1}^{m/2-1} d^i + \sum_{i=1}^{m/2-1} d^i \leqslant 2 \sum_{i=1}^{\infty} d^i. \end{aligned}$$

Accordingly we treat the case where m is an odd integer.

As a consequence of Lemma 4(b) we get in the same way for every $m \in \mathbb{N}$

$$\left(\sum_{k=2}^{\infty} x^{n_{m+k}}/u_{n_{m+k}} \right) \Big/ P_1(x) \leqslant 2 \sum_{i=1}^{\infty} d^i. \tag{31}$$

Using (30) and (31) we obtain for $m \geqslant 3$ and $x \in [v_{n_{m-1}}, v_{n_m})$

$$\begin{aligned} &\left(\sum_{k=0}^{\infty} x^{n_k}/u_{n_k} \right) \Big/ P_1(x) \\ &\quad = \left(\sum_{k=3}^{m} x^{n_{m-k}}/u_{n_{m-k}} + \sum_{k=1}^{4} x^{n_{m-3+k}}/u_{n_{m-3+k}} + \sum_{k=2}^{\infty} x^{n_{m+k}}/u_{n_{m+k}} \right) \Big/ P_1(x) \\ &\quad \leqslant 2 \sum_{i=1}^{\infty} d^i + 4 + 2 \sum_{i=1}^{\infty} d^i = 4/(1-d). \end{aligned}$$

Using only (30) we get the above estimation for $m = 2$ and $m = 1$. Finally in the interval $x \in [0, v_0)$ we have $P_1(x) = 1/u_0 = P_1(v_0)$, and therefore the inequality

$$1 \leqslant \left(\sum_{k=0}^{\infty} x^{n_k}/u_{n_k} \right) \Big/ P_1(x) \leqslant 4/(1-d) \tag{32}$$

holds for every $x \geqslant 0$.

We have already mentioned (proof of Theorem 2, (8)) that for $x \in [v_{n_{m-1}}, v_{n_m})$ the function P has the representation

$$P(x) = x^i/u_i, \qquad \text{for} \quad x \in [v_{i-1}, v_i), \tag{33}$$

where $n_{m-1} < i \leqslant n_m$.

With the aid of Lemma 1 we get for $m \in \mathbb{N}$, $x \leqslant v_{n_{m-1}}$ and $i > n_{m-1}$

$$v_{n_{m-1}}^i/u_i \leqslant v_{n_{m-1}}^{n_{m-1}}/u_{n_{m-1}}, \tag{34}$$

and for $x = v_{n_m}$ and $i \leqslant n_m$

$$v_{n_m}^i/u_i \leqslant v_{n_m}^{n_m}/u_{n_m}. \tag{35}$$

Based on our definition of the subsequence $\{u_{n_k}, k \in \mathbb{N}_0\}$ we have for every $k \in \mathbb{N}_0$ that at least one of the inequalities (25), (26) holds. Hence for every $m \in \mathbb{N}$ at least one of the following estimations (36), (37) holds:

$$u_{n_{m-1}} v_{n_{m-1}}^{n_m - n_{m-1}}/u_{n_m} = A(n_{m-1}, s_{m-1} - 1) \geqslant d, \tag{36}$$

$$u_{n_m} v_{n_m}^{n_{m-1} - n_m}/u_{n_{m-1}} = B(n_{m-1}, s_{m-1} - 1) \geqslant d. \tag{37}$$

First let us assume that (36) holds. Then for $x \in [v_{n_{m-1}}, v_{n_m})$ we get by (29), (33), and (34)

$$\begin{aligned} P_1(x)/P(x) &= P_1(x)/(x^i/u_i) \geqslant (x^{n_m}/u_{n_m})/(x^i/u_i) \geqslant (v_{n_{m-1}}^{n_m}/u_{n_m})/(v_{n_{m-1}}^i/u_i) \\ &\geqslant (v_{n_{m-1}}^{n_m}/u_{n_m})/(v_{n_{m-1}}^{n_{m-1}}/u_{n_{m-1}}) = u_{n_{m-1}} v_{n_{m-1}}^{n_m - n_{m-1}}/u_{n_m} \geqslant d. \end{aligned}$$

Now let us assume that (37) holds. Then we get by (29), (33), and (35)

$$\begin{aligned} P_1(x)/P(x) &= P_1(x)/(x^i/u_i) \geqslant (x^{n_{m-1}}/u_{n_{m-1}})/(x^i/u_i) \geqslant (v_{n_m}^{n_{m-1}}/u_{n_{m-1}})/(v_{n_m}^i/u_i) \\ &\geqslant (v_{n_m}^{n_{m-1}}/u_{n_{m-1}})/(v_{n_m}^{n_m}/u_{n_m}) = u_{n_m} v_{n_m}^{n_{m-1} - n_m}/u_{n_{m-1}} \geqslant d. \end{aligned}$$

In both cases, (36) and (37), we get for $m \in \mathbb{N}$ and $x \in [v_{n_{m-1}}, v_{n_m})$

$$P_1(x)/P(x) \geqslant d.$$

For $x \in [0, v_0)$ we have $P_1(x) = P(x) = 1/u_0$. Hence it follows that

$$1 \geqslant P_1(x)/P(x) \geqslant d, \qquad \text{for} \quad x \geqslant 0. \tag{38}$$

Comparing (32), (38), and our assumption (3) we finally obtain

$$(1-d)/4 \leqslant h(x)/\sum_{k=0}^{\infty} x^{n_k}/u_{n_k} \leqslant c/d, \qquad \text{for} \quad x \geqslant 0.$$

Remarks. (a) In [4] Erdős and Kövari construct a power series $N(x)=\sum c_n x^n$, $c_n \geqslant 0$, with $M \sim N$, where $M(x)$ is the maximum modulus of an entire function $f(z)$, $z \in \mathbb{C}$. Their construction is mainly based on the convexity of the function $F(t)=\ln M(e^t)$ and the fact that $M(x)$ is the maximum modulus of an entire function.

The construction method shown as proof of Theorem 4 for the power series $N(x)=\sum c_n x^n$, $c_n \geqslant 0$, with $h \sim N$, gets rid of all additional requirements on $h(x)$. We only assume that $h(x)$ grows faster than any power of x at infinity, to ensure that $N(x)$ does not represent a polynomial.

(b) Very often even power series are of special importance (see for example page 9). Regarding $h(\sqrt{x})$ instead of $h(x)$ in Theorem 4 and 1, respectively, we get an even power series $N(x)=\sum c_n x^{2n}$, $c_n \geqslant 0$, with $h \sim N$.

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